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NOTE

# CYCLES THROUGH $k$ VERTICES IN BIPARTITE TOURNAMENTS*

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We give a simple proof that every $k$-connected bipartite tournament has a cycle through every set of $k$ vertices. This was conjectured in [4].

We shall assume that the reader is familiar with the standard terminology on graphs and digraphs and refer the reader to [2].

A digraph $D$ is *strong* if it contains a directed path from $x$ to $y$ for every choice of distinct vertices $x,y$. $D$ is *$k$-connected* if it has a $k$ internally disjoint directed paths from $x$ to $y$ for every choice of distinct vertices $x,y$. If there is no arc from $x$ to $y$, then by Menger's theorem this equivalent to saying that every set of vertices $S \in V(D) - \{x,y\}$, whose removal destroys all paths from $x$ to $y$, contains at least $k$ vertices.

A *factor* of a digraph $D$ is a spanning subdigraph such that every vertex has in and out-degree 1, i.e. a collection of disjoint cycles covering the vertices of $D$. The existence of a factor can be checked, and the factor found if it exists, in any digraph in time $O(n^{\frac{5}{2}})$, where $n$ is the number of vertices of $D$, e.g. see [5] pp. 72–73.

It is well-known that every strong tournament has a hamiltonian cycle and thus also a cycle through any subset of its vertex set. For bipartite tournaments (i.e. orientations of complete bipartite graphs) the situation is quite different. There exists arbitrarily highly connected bipartite tournaments with no hamiltonian cycle, and there exist ($k$–1)-connected bipartite tournaments with $k$ specific vertices which lie on no common cycle [4].

In fact for bipartite tournaments, it is the presence of a factor that is important for the hamiltonian cycle problem :

**Theorem 1.** [4] *A strong bipartite tournament has a hamiltonian cycle if and only if it has a factor.*

Given any factor one can also find a hamiltonian cycle fast.

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**Theorem 2.** [6] *Given a factor $C'_1,\dots,C'_r$, $r\geq 1$, of a bipartite tournament $B$ on $n$ vertices, one can find, in time $O(n^2)$, another factor $C_1,\dots,C_k$, $1\leq k\leq r$, such that there is no arc from $C_j$ to $C_i$ for $j>i$.*

We call a digraph $D$ $k$-cyclic if it contains a cycle through every subset of $k$ vertices of $V(D)$. In [4] examples of $(k-1)$-connected non-$k$-cyclic bipartite tournaments were given, it was shown that every 2-connected bipartite tournament is 2-cyclic and it was conjectured that for all integers $k\geq 1$ every $k$-connected bipartite tournament is $k$-cyclic. Below we give a simple proof of that conjecture.

**Theorem 3.** *If $D$ is a $k$-connected bipartite tournament, $k\geq 1$, then $D$ is $k$-cyclic.*

**Proof.** Clearly it is enough to prove that $D$ has a cycle through every set of $k$ distinct vertices. Let $Z=\{x_1,\dots,x_k\}$ be an arbitrary set of $k$ distinct vertices of $D$. We shall show that $D$ has a cycle containing all of these vertices.

We first prove that $D$ has a collection of disjoint cycles covering all the vertices in $Z$. Using Hall's theorem, we conclude from the $k$-connectivity of $D$ that $V(D)-Z$ contains $k$ distinct vertices $y_1,\dots,y_k$ such that there is an arc from $y_i$ to $z_i$ for $i=1,2,\dots,k$. By a version of Menger's theorem $D$ contains $k$ disjoint paths from $Z$ to $\{y_1,\dots,y_k\}$. The subgraph consisting of these paths and the arcs $y_1\to z_1, y_2\to z_2,\dots,y_k\to z_k$ is a union of disjoint cycles $C_1,\dots,C_t$, $t\geq 1$ covering $Z$.

If $t=1$ then we have the desired cycle, so we may assume, by Theorem 2, that $t\geq 2$ and no cycle $C_i$ contains all the vertices of $Z$ and that there are no arcs from $C_j$ to $C_i$ for any $i,j$ such that $j>i$. Note that if $P$ is any path starting at a vertex $u\in V(C_t)$ and ending in a vertex $v\in V(C_i)$, $1\leq i\leq t-1$, such that $P$ has only the vertices $u,v$ in common with $V(C_1)\cup\dots\cup V(C_t)$, then $D$ has a cycle $C$ with vertex set $V(C_i)\cup\dots\cup V(C_t)\cup V(P)$ if $u$ and $v$ are adjacent or, if $u$ and $v$ are non-adjacent, a cycle $C'$ with vertex set $V(C_i)\cup\dots\cup V(C_t)\cup V(P)-w$, where $w$ can be chosen to be either the successor of $u$ on $C_t$ or the predecessor of $v$ on $C_i$.

Suppose first that $V(C_t)-Z\neq\emptyset$. Let $S$ be the set of predecessors of vertices in $Z\cap V(C_t)$. Note that $|S|<k$, since $t\geq 2$. Since $D$ is $k$-connected and $C_t$ contains a vertex not in $Z$, there exists a path $P$ from a vertex $u\in V(C_t)-S$ to a vertex $v\in V(C_1)\cup\dots\cup V(C_{t-1})$. If $u$ and $v$ are adjacent, then, as we argued above, $D$ contains a cycle with vertex set $V(C_i)\cup\dots\cup V(C_t)\cup V(P)$, where $v\in V(C_i)$. Otherwise $D$ contains a cycle with vertex set $V(C_i)\cup\dots\cup V(C_t)\cup V(P)-w$, where $w$ is the successor of $u$ on $C_t$. In both cases we obtain a smaller collection of cycles containing all the vertices of $Z$. So we may assume that $V(C_t)\subset Z$.

If $V(C_1)\cup\dots\cup V(C_{t-1})\not\subset Z$ then, by the $k$-connectivity of $D$ and the fact that $C_1\cup\dots\cup C_{t-1}$ contain less than $k$ vertices of $Z$, we can find a path $P$ from $C_t$ to $V(C_1)\cup\dots\cup V(C_{t-1})$ which ends in a vertex $v\in V(C_i)$, for some $1\leq i\leq t-1$, whose predecessor on $C_i$ does not belong to $Z$. Now we can see as above how to obtain a smaller collection of cycles containing the vertices of $Z$.

Suppose now that $V(C_1)\cup\dots\cup V(C_t)=Z$. By the $k$-connectivity of $D$, there exists a $(u,v)$-path $P$, only intersecting $V(C_1)\cup\dots\cup V(C_t)$ in $u$ and $v$, for every choice of distinct vertices $u\in V(C_t)$, $v\in V(C_1)\cup\dots\cup V(C_{t-1})$. In particular we can choose $u,v$ such that they are adjacent. Thus again we get a smaller collection of cycles containing all the vertices of $Z$.

We have shown above that given a collection of at least two disjoint cycles covering $Z$ we can always find a smaller collection of cycles with the same property and the proof is complete. ∎

**Corollary 4.** *There exits a polynomial algorithm to find a cycle through any set $Z$ of $k$ vertices in a $k$-connected bipartite tournament.* ∎

Note that for general digraphs there is no sufficient condition in terms of connectivity that will imply the existence of a cycle through two given vertices $x$ and $y$. This was shown in [7]. Also for general digraphs the problem of deciding the existence of a cycle through $k$ given vertices is NP-complete for any fixed $k \geq 2$ [3].

An obvious necessary condition for the existence of a cycle though a set $Z$ of $k$ given vertices is that there exists a collection $C_1, \dots, C_t$, $t \geq 1$ of disjoint cycles covering $Z$. We start our proof above by proving how $k$-connectivity is sufficient to guarantee the existence of these cycles for bipartite tournaments. The $(k-1)$-connected non-$k$-cyclic example in [4] does not satisfy this condition. Thus one might ask if even just strongly connected is sufficient together with the existence of the cycles covering $Z$. It is not difficult to find examples showing that this is not the case, but we did not find any 2-connected such examples.

It is not difficult to see that one can check the existence of the cycles $C_1, \dots, C_t$, $t \geq 1$ above in polynomial time. In this light we conjecture the following.

**Conjecture 5.** *For fixed $k$ there exists a polynomial algorithm to decide the existence of a cycle through given set of $k$ vertices in a bipartite tournament and find one if it exists.*

One might also ask a similar question for cycles through a set of $k$ disjoint arcs. Here we pose the following conjecture.

**Conjecture 6.** *For any fixed natural number $k$ there exists a polynomial algorithm to decide the existence of a cycle through $k$ given arcs in a bipartite tournament and to find one if it exists.*

Note that even the case $k=2$ is open and seems nontrivial. The corresponding conjecture for tournaments has been verified in the case $k=2$ in [1]. The algorithm given there is quite complicated. If $k$ is not fixed, but is part of the input, it is NP-complete to determine the existence such a cycle. This follows from the proof of Theorem 6.1 in [1].

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